Stability of a mass accreting shell expanding in a plasma

S. V. Bulanov[1] and F. Pegoraro[2]
[1]*General Physics Institute, Russian Academy of Sciences, Vavilov Street 38, 119991 Moscow, Russia*
[2]*Department of Physics and INFM, Pisa University, via Buonarroti 2, Pisa, Italy*
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A linearized analysis is presented of the stability of a shell which accretes mass as it expands in a plasma under the push of the electromagnetic radiation trapped inside it. The interaction with the radiation is described in terms of a ponderomotive force and the shell dynamics is treated within the snowplow approximation. The mass accretion and the radiation expansion are shown to affect the stability of planar, cylindrical, and spherical shells differently.

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I. INTRODUCTION

The expansion of plasma bubbles and cavities in an ambient plasma has been widely studied both in the laboratory and in the astrophysical context with reference to different physical phenomena, ranging from the expansion of supernova remnants to the evolution of relativistic electromagnetic solitons. The free expansion of a spherical bubble of cold plasma and its retardation due to the effect of the external material can be described with the exact self-similar solution obtained by Sedov [1] and Taylor [2]. However, this solution implies perfect spherical symmetry and cannot be used when the symmetry is broken. In this case the snowplow approximation provides a convenient tool for a qualitative and quantitative description of the expansion as was shown more than 40 years ago. The snowplow model was developed in Ref. [3] in application to the theory of the plasma focus. In Ref. [4] it was shown how to solve the Sedov-Taylor problem within the snowplow approximation. The bubble expansion was also considered within the framework of the snowplow approximation in Ref. [5] in the context of the study of supernovae explosions. In the snowplow model [6] the ambient matter is assumed to collide inelastically with an infinitely thin shell of plasma expanding in a medium. The shell sweeps through space and piles up the matter it encounters on its surface, just as the blade of a snowplow collects the snow.

More detailed models of the supernovae expansion were later developed, see, e.g., Refs. [7] and references therein, and the problem of the stability of the spherical expansion was addressed in the case of an inhomogeneous medium in Ref. [8] and numerically in Ref. [9], while in Ref. [10] it was recognized that the mass accretion in the snowplow mechanism tends to suppress the development of the Rayleigh-Taylor instability. The acceleration of charged particles due to the expansion of these spherical plasma shells was studied in Ref. [11]. We recall that the acceleration of charged particles at the front of shock waves provides one of the most important acceleration mechanisms which has been under discussion for many years (see, for example, Refs. [6] and [12] and literature quoted therein). In this context, the studies of the shock wave sphericization is of great importance for the problem of cosmic ray acceleration by the curved shock waves [13,14].

Recently it has been recognized that the physics of the interaction of ultraintense laser pulses with plasmas provides the opportunity for simulating such cosmic events as a supernova explosion in the laboratory [15]. In addition, in the interaction of a laser pulse with a plasma, a phenomenon has been identified [16] that directly involves the expansion of a cavity filled with electromagnetic energy, its interaction, and coalescence with other cavities in the plasma and the eventual sphericization of the resulting structure. This phenomenon is related to the long time evolution of the relativistic subcycle electromagnetic solitons [17,18] that are produced when an ultrashort ultraintense laser pulse propagates in an underdense plasma. Interestingly, as in the cosmic case, these small scale structures have been considered as a mechanism of particle acceleration [19].

Relativistic subcycle solitons [17] consists of slowly or nonpropagating electron density cavities inside which an electromagnetic field is trapped and oscillates coherently with a frequency below the local plasma frequency and a spatial structure corresponding to half a cycle. These nonlinear structures are commonly observed in one- and two-dimensional (1D and 2D) particle in cell (PIC) simulations of the laser pulse interaction with a plasma. Recently, they have been identified in three-dimensional (3D) PIC simulations [18] and their long term effect on the plasma has been proposed in Ref. [20] in order to explain the experimental detection with the technique of proton imaging of long lived electric field bubbles in a plasma after the interaction with an ultraintense laser pulse.

As discussed in Ref. [16], in an electron-ion plasma subcycle solitons evolve on the ion dynamical time scale into postsolitons, i.e., into slowly expanding quasineutral cavities filled by electromagnetic radiation. This evolution is caused by the ion acceleration due the time-averaged electrostatic field inside the soliton. In Ref. [16] the theory of these expanding quasineutral cavities was developed within the framework of the snowplow model. In this framework the plasma cavity forms a resonator for the trapped electromagnetic field and expands under the push of its ponderomotive force exerted by the electromagnetic fields. As the cavity expands, the amplitude and the frequency of the electromagnetic field E decrease. In the adiabatic limit the ratio $\int \mathbf{E}^2 dV/\omega_s$ between the energy and the frequency remains constant so that $E^2 \sim r^{-4}$ with r the cavity radius. Under the action of the electromagnetic pressure the walls of the cavity move, piling up plasma as a snowplow. The mass inside the expanding plasma shell is equal to the mass $M = 4\pi n_0 m_i r^3/3$ initially contained inside a sphere of the radius r and the momentum conservation equation gives

$$d[M(dr/dt)]/dt = 4\pi r^2 \langle \mathbf{E}^2 \rangle / 8\pi, \tag{1}$$

which yields [16] for the shell radius $r=r(t)$ in dimensionless units

$$d[r^3(dr/dt)]/dt = r^{-2}. \tag{2}$$

Asymptotically for $t \to \infty$ the postsoliton radius increases as $r \sim t^{1/3}$.

In Ref. [20] simulations of the two postsolitons and of multisoliton merging were performed with the aim of interpreting the experimental detection, reported in the same paper, of electric field bubbles. In these simulations the structures that result from the merging evolve towards a nearly spherical shape, i.e., the initial nonspherical perturbation decays. This implies that the expansion of spherical postsolitons is stable.

In the present paper we examine the linear stability of postsolitons analytically within the framework of the snowplow approximation in the limit where the shell walls are infinitely thin. We find that the bubbles are stable against a modified form of the Rayleigh-Taylor instability up to relatively short wave numbers or where the shell model can no longer be expected to be valid. These short wavelength modulations are not seen in the numerical simulations of the expanding postsolitons in Ref. [20] and are likely to be stabilized by the effects of the finite width of the shell not included in the snowplow approximation.

The aim of the present paper is to provide a general treatment of the linear stability of a thin shell that accretes in the snowplow approximation. Therefore we examine different shell geometries corresponding to the expansion of a planar, of a cylindrical, and of a spherical shell. We find that the faster decrease of the ponderomotive force and the higher mass intake that characterize the spherical expansion lead to increased stability. As a byproduct of the present analysis we derive the time evolution of the deformation and of the motion of a postsoliton expanding in a weakly inhomogeneous plasma. An interesting extension of the present work will consist of the inclusion of a "rocket" term in the equations for the mass evolution and for the momentum conservation of the shell so as to analyze the effect on the shell stability of the bouncing back of the material that is accreted on the shell surface and the loss of mass left behind inside the expanding cavity.

II. GOVERNING EQUATIONS

We adopt the set of equations that has been used for the description of the Rayleigh-Taylor instability of a thin shell (see Ref. [21]), modified in order to account for the increase of the shell mass and the decrease of the push of the electromagnetic pressure as the shell expands and sweeps the background plasma.

The mass conservation equation can be written in Lagrangian variables as

$$\partial_t(\sigma\ d\Sigma\ \mathbf{u}) = p(\mathbf{r})\mathbf{d\Sigma}, \tag{3}$$

where $\mathbf{r}(t)$ is the position of the shell element at time t. Here the surface density σ, the surface element $\mathbf{d\Sigma} = d\Sigma\ \mathbf{n}$, the shell velocity $\mathbf{u}$, and normal unit vector $\mathbf{n}$ are functions of $\mathbf{r}(t)$. Analogously, the momentum conservation equation can be written as

$$\partial_t(\sigma d\Sigma) = \rho_0 \mathbf{u} \cdot \mathbf{d\Sigma}, \tag{4}$$

with ρ_0 the external density (assumed constant and uniform for the sake of simplicity) that is being accreted and $p(\mathbf{r})$ the ponderomotive pressure that pushes the shell walls due to the trapped electromagnetic radiation. We close this system of equations by adopting the following model for the ponderomotive pressure term $p(\mathbf{r})$:

$$p(\mathbf{r})/p_0 = (|\mathbf{r}_0|/|\mathbf{r}|)^{1+D}. \tag{5}$$

Here D is the dimension of the expansion process: $D=1$ in a planar geometry, $D=2$ in a two-dimensional (cylindrica) expansion, and $D=3$ in the three-dimensional (spherical) case. Integrating Eq. (4) we obtain

$$\sigma(d\Sigma/d\Sigma_0) = \sigma_0 + \rho_0 \int_{t_0}^{t} \mathbf{u} \cdot \mathbf{d\Sigma}/d\Sigma_0 \equiv m(t), \tag{6}$$

where $d\Sigma_0 = d\alpha \times d\beta$ is the Lagrangian surface element, α and β are the Lagrangian variables chosen such that at $t = t_0$ we have $\sigma = \sigma_0$ and $d\Sigma = d\Sigma_0$, and $m(t)$ is the shell mass for unit initial surface area. For sufficiently large times the contribution of the initial density σ_0 to the shell mass can be neglected.

Introducing dimensionless units with the density normalized on $\rho_0 r_0$, the velocity on $(p_0/\rho_0)^{1/2}$, lengths on r_0, and time on $r_0(p_0/\rho_0)^{-1/2}$, Eqs. (3) and (4) take the form

$$\partial_t m = (\partial_t \mathbf{r} \cdot [\partial_\alpha \mathbf{r} \times \partial_\beta \mathbf{r}]), \tag{7}$$

$$\partial_t(m \partial_t \mathbf{r}) = |\mathbf{r}|^{-(1+D)}[\partial_\alpha \mathbf{r} \times \partial_\beta \mathbf{r}]. \tag{8}$$

III. SHELL EXPANSION IN A PLANE

In this section we first determine the steady expansion rate of a planar shell (i.e., of a shell that expands along x and that at $t=0$ is located at the $x=0$ line) and of a cylindrical shell (i.e., of a shell that expands radially and that at $t=0$ is located at $R=R_0$ with R the distance from the origin in the x-y plane) in the two-dimensional x-y plane. Subsequently, we investigate the linear stability of this expansion. In terms of Eq. (5) the planar case, where the pressure decreases on $|x|^{-2}$, corresponds to $D=1$ and the cylindrical case, where the pressure depends on R^{-3}, to $D=2$.

Following Ref. [21] we find it convenient to introduce the complex variable

$$w = x + iy, \tag{9}$$

so that

$$x=(w+w^*)/2, \quad y=-i(w-w^*)/2$$

and (10)

$$R^2=ww^*,$$

with the asterisk denoting complex conjugate. Then Eqs. (7) and (8) can be written as

$$\partial_t m=\frac{i}{2}(\partial_t w\partial_\alpha w^*-\partial_\alpha w\partial_t w^*), \tag{11}$$

$$\partial_t(m\partial_t w)=-ip(w,w^*)\partial_\alpha w, \tag{12}$$

where

$$p(w,w^*)=\frac{4}{(w+w^*)^2} \quad \text{and} \quad p(w,w^*)=\frac{1}{(ww^*)^{3/2}} \tag{13}$$

in the planar and in the cylindrical case, respectively.

A. Planar shell expansion

The steady expansion of a planar shell is obtained by defining the Lagrangian variable α as the Cartesian coordinate along the shell at $t=0$ and by setting

$$w_0(t,\alpha)=x_0(t)+i\alpha, \quad \text{i.e.,} \quad y_0(t)=y_0(t=0)\equiv\alpha. \tag{14}$$

Then from Eqs. (12) and (13), or directly from Eqs. (7) and (8), we obtain

$$m_0(t)=x_0(t) \quad \text{and} \quad \partial_{tt}x_0(t)^2=2/x_0(t)^2, \tag{15}$$

which, for $t\gg1$, gives

$$x_0(t)\approx[(2t)^2\ln(t)]^{1/4}. \tag{16}$$

The kinetic energy of the planar shell decreases with time as $(t)^{-1/2}[\ln(t)]^{3/4}$.

B. Cylindrical shell expansion

The steady expansion of a cylindrical shell is obtained by defining the Lagrangian variable α as the azimuthal angle along the shell at $t=0$ and by setting

$$w_0(t,\alpha)=R_0(t)\exp(i\alpha), \tag{17}$$

i.e.,

$$x_0(t)=R_0(t)\cos(\alpha), \quad y_0(t)=R_0(t)\sin(\alpha). \tag{18}$$

Then from Eqs. (12) and (13), or directly from Eqs. (7) and (8), we obtain

$$m_0(t)=R_0^2(t)/2 \quad \text{and} \quad R_0(t)=(5t)^{2/5}. \tag{19}$$

The kinetic energy of the cylindrical shell decreases with time as $t^{-2/5}$.

C. Planar shell expansion stability

We linearize Eqs. (12) and (13) around the slab solution w_0 given by Eq. (14). Note that for a planar slab, Eqs. (12) and (13), together with the equilibrium solution w_0, are translationally invariant along α (i.e., along y) and are invariant under the transformation

$$w\leftrightarrow w^*, \quad \alpha\leftrightarrow-\alpha \tag{20}$$

[in the case of Eq. (13) we obtain its complex conjugate equation]. We set

$$w(\alpha,t)=w_0+\delta w=w_0+\int_{-\infty}^{+\infty}dk\, w_k\exp(ik\alpha) \tag{21}$$

and see that Eqs. (12) and (13) couple the k and $-k$ harmonics. Using the invariance properties of Eqs. (12) and (13) we find (see Appendix A) that for a chosen $|k|$ $(=k)$, i.e., disregarding the relative phase between modes with different $|k|$, we can take w_k and w_{-k} to be real. Then we write

$$m(\alpha,t)=m_0(t)+\delta m_k(\alpha,t), \tag{22}$$

where the $|k|$ component δm_k of the mass variation can be written in terms of its amplitude $\mu_k(t)$ and of the variable α as

$$\begin{aligned}\delta m_k(\alpha,t)&=\mu_k(t)[\exp(ik\alpha)+\exp(-ik\alpha)]\\&=2\mu_k(t)\cos(k\alpha),\end{aligned} \tag{23}$$

while the perturbed pressure term $4/(w+w^*)^2$ gives

$$4/(w+w^*)^2\approx[1-2\cos(k\alpha)(w_k+w_{-k})/x_0]/x_0^2. \tag{24}$$

Thus, we obtain the following coupled ordinary differential equations:

$$\begin{aligned}\partial_t\mu_k(t)&=k\dot{x}_0(w_k-w_{-k})/2+(\dot{w}_k+\dot{w}_{-k})/2\\&=(x_0)^{-k}\partial_t[(x_0)^k w_k]/2+(x_0)^k\partial_t[(x_0)^{-k}w_{-k}]/2\end{aligned} \tag{25}$$

and

$$\partial_t(m_0(t)\partial_t w_k)+\partial_t(\mu_k\partial_t x_0)=\frac{kw_k}{x_0^2}-\frac{w_k+w_{-k}}{x_0^3}, \tag{26}$$

$$\partial_t(m_0(t)\partial_t w_{-k})+\partial_t(\mu_k\partial_t x_0)=\frac{-kw_{-k}}{x_0^2}-\frac{w_k+w_{-k}}{x_0^3}. \tag{27}$$

For very large t such that $kx_0\gg1$, the last terms on the right-hand side (rhs) of Eqs. (26) and (27), which represent the effect of the perturbed pressure, can be neglected and (disregarding unimportant logarithmic terms) we can take

$$\begin{aligned}\mu_k(t)&=\hat{\mu}_k m_0(t)\exp(\gamma t^{1/4}),\\w_k(t)&=\hat{w}_k x_0(t)\exp(\gamma t^{1/4}),\\w_{-k}(t)&=\hat{w}_{-k}x_0(t)\exp(\gamma t^{1/4}),\end{aligned} \tag{28}$$

where $\gamma=\gamma(|k|)$ and $m_0(t)\approx x_o(t)\approx 8^{1/4}t^{1/2}$. Stability corresponds to nonpositive values of γ. However, it is easy to see that the second term on the left-hand side (lhs) of Eqs. (26) and (27) is smaller than the first for $t \gg |\gamma|^{-1/4}$ so that the mass term perturbation can be neglected. Then we obtain

$$\gamma^2=\pm k, \tag{29}$$

as in the Ott problem [22] discussed in Ref. [21]. This result shows that the asymptotic stability properties of a planar shell expansion are not improved in an essential way by the mass accretion and by the decrease of the pressure term that simply result in the slowdown [see Eq. (28)] of the growth of the Rayleigh-Taylor instability without, however, affecting the instability conditions.

The fifth solution of the system of Eqs. (25)–(27) corresponds to setting

$$w_{-k}(t)=-w_{-k}(t)=\text{const}, \quad \text{i.e,} \quad \delta x=0 \tag{30}$$

and is independent of k. This solution has no physical relevance and corresponds to a relabeling of the variable $y=y(\alpha)$ in the equilibrium solution in Eq. (14) from $y=\alpha$ to $y=f(\alpha)$ and the corresponding change in $m_0(t)$ to $m_0(t)\ [df(\alpha)/d\alpha]$.

D. Cylindrical shell expansion stability

We linearize Eqs. (12) and (13) around the cylindrical solution w_0 given by Eq. (17) and write

$$\begin{aligned} w(\alpha,t)&=w_0+\delta w \\ &=R_0(t)\exp(i\alpha)+\Sigma_s R_s(t)\exp[i(s+1)\alpha], \end{aligned} \tag{31}$$

where δw has been expanded in a Fourier series and the sum extends from $-\infty$ to $+\infty$. As in the planar case the Fourier modes s and $-s$ are coupled and we can take $R_s(t)$ to be real. We label the pairs of coupled modes by the index s and write

$$m(\alpha,t)=m_0(t)+\delta m_s(\alpha,t), \tag{32}$$

where

$$\begin{aligned} \delta m_s(\alpha,t)&=\mu_s(t)[\exp(is\alpha)+\exp(-is\alpha)] \\ &=2\mu_s(t)\cos(s\alpha), \end{aligned} \tag{33}$$

while the perturbed pressure term $(ww^*)^{-3/2}$ gives

$$(ww^*)^{-3/2}\approx R_0^{-3}-3[(R_s+R_{-s})/R_0^3]\cos(s\alpha). \tag{34}$$

Then, analogously to the planar case, we obtain the following coupled ordinary differential equations:

$$\partial_t\mu_s=\frac{1}{2}\left[\frac{\partial_t(R_0^{(1+s)}R_s)}{R_0^s}+\frac{\partial_t(R_0^{(1-s)}R_{-s})}{R_0^{-s}}\right] \tag{35}$$

and

$$\partial_t[m_0(t)\partial_t R_s]+\partial_t(\mu_s\partial_t R_0)=\frac{(1+s)R_s}{R_0^3}-\frac{3(R_s+R_{-s})}{2R_0^3}, \tag{36}$$

$$\partial_t[m_0(t)\partial_t R_{-s}]+\partial_t(\mu_s\partial_t R_0)=\frac{(1-s)R_{-s}}{R_0^3}-\frac{3(R_s+R_{-s})}{2R_0^3}. \tag{37}$$

The solutions of Eqs. (35)–(37) are of the form

$$\mu_s(t)=\hat{\mu}_s t^{\gamma+2/5}, \quad R_s(t)=\hat{R}_s t^{\gamma}, \quad R_{-s}(t)=\hat{R}_{-s}t^{\gamma}, \tag{38}$$

where the exponent $\gamma=\gamma(s)$ is given by the roots of the fourth-order polynomial

$$12s^2+5\gamma(6+55\gamma+150\gamma^2+125\gamma^3)=0. \tag{39}$$

Stable solutions correspond to Re $\gamma\leqslant 2/5$. We find stability for $s\leqslant 6$. The fifth solution of Eqs. (35)–(37) corresponds to $\gamma=2/5$ and has to $R_s=-R_{-s}$ for all s which implies $\delta(ww^*)=0$, i.e., the distance from the origin is not changed. As in the planar solution this fifth solution corresponds to a relabeling of the azimuthal angle in the equilibrium solution as a function of the Lagrangian variable α. For large values of s Eq. (39) reduces to $\gamma^2=\pm i(2\sqrt{3}/25)s$ which is a modification of Ott's result [22]. These results show that the effect of the mass accretion and of the decrease of the pressure stabilizes the long wavelength Rayleigh-Taylor modes of a cylindrical foil. The instabilities at short wavelengths (large s) can be expected to be stabilized by effects related to the finite width of the expanding shell which are not accounted for in the snowplow model.

E. Motion in a weakly inhomogeneous background

A perturbation approach analogous to the one used for the above stability analysis makes it possible to calculate the motion and the deformation of a cylindrical shell in a weakly inhomogeneous background.

We consider a weakly inhomogeneous 2D configuration with an external plasma density of the form

$$\rho_0(\mathbf{x})=\rho_0[1+\epsilon(v^*w+vw^*)], \tag{40}$$

with $\epsilon\ll 1$ and v a complex number such that $vv^*=1$. To leading order in ϵ we can use Eqs. (36) and (37) with $s=1$, together with the mass equation (35) where the effect of the plasma inhomogeneity takes the form of a forcing term. Choosing the Lagrangian angle α such that $v=1$ (i.e., taking the plasma density to vary along x) instead of Eq. (35) we obtain

$$\partial_t\mu_1=R_0[2\partial_t R_1+\partial_t R_{-1}]/2+\epsilon R_0\partial_t m_0(t). \tag{41}$$

For large t the above system of equations reduces to

$$\partial_t\mu_1\approx\epsilon R_0\partial_t m_0(t) \tag{42}$$

and

$$\partial_t[m_0(t)\partial_t R_1]\approx-\partial_t(\mu_1\partial_t R_0), \tag{43}$$

$$\partial_t[m_0(t)\partial_t R_-]\approx-\partial_t(\mu_1\partial_t R_0), \tag{44}$$

which give

$$R_1\approx R_{-1}\propto t^{4/5},\quad R_1<0,\ R_{-1}<0 \tag{45}$$

and show that the shell negative shift along x and its deformation grow together and faster than the equilibrium expansion.

IV. SHELL EXPANSION IN SPACE

The expansion of a three-dimensional shell and its stability properties are intrinsically different from those of a shell expanding in a plane. As explicitly discussed in Ref. [23], the equations of the shell expansion in space are nonlinear in the Lagrangian variables α and β, as shown by Eqs. (7) and (8), while in the case of the expansion in a plane the only nonlinearities arise from $m(t)$ and $p(t)$.

We consider a spherical configuration with

$$\alpha=\cos\theta,\quad \beta=\varphi, \tag{46}$$

where θ and φ are the usual spherical angular coordinates (in Lagrangian variable space) and take $D=3$. Thus the pressure given by Eq. (8) has the form $p=1/|r|^4$.

Because of the additional nonlinearity in the Lagrangian variables we are not able to introduce a representation that generalizes Eqs. (11)–(13) and thus we examine the spherical shell expansion and its stability separately.

A. Spherical shell expansion

The unperturbed spherical shell expansion is described by

$$\begin{gathered} x_0(\alpha,\beta,t)=r_0(t)(1-\alpha^2)^{1/2}\sin\beta,\\ y_0(\alpha,\beta,t)=r_0(t)(1-\alpha^2)^{1/2}\cos\beta,\\ z_0(\alpha,t)=r_0(t)\alpha. \end{gathered} \tag{47}$$

In order to determine the time dependence of the radius $r_0(t)$ and in Sec. IV B the expansion stability, a number of geometrical relationships turn out to be useful. From Eq. (47) we obtain

$$\begin{gathered} \partial_\alpha x_0=-r_0(t)\alpha(1-\alpha^2)^{-1/2}\sin\beta,\\ \partial_\beta x_0=r_0(t)(1-\alpha^2)^{1/2}\cos\beta,\\ \partial_\alpha y_0=-r_0(t)\alpha(1-\alpha^2)^{-1/2}\cos\beta,\\ \partial_\beta y_0=-r_0(t)(1-\alpha^2)^{1/2}\sin\beta.\\ \partial_\alpha z_0=r_0(t),\quad \partial_\beta z_0=0. \end{gathered} \tag{48}$$

We see that $\partial_t\mathbf{r}_0$ is parallel to $\mathbf{r}_0$ and that

$$\partial_\alpha\mathbf{r}_0\cdot\partial_\beta\mathbf{r}_0=\partial_\alpha\mathbf{r}_0\cdot\partial_t\mathbf{r}_0=\partial_\beta\mathbf{r}_0\cdot\partial_t\mathbf{r}_0=0, \tag{49}$$

i.e., $\partial_\alpha\mathbf{r}_0,\ \partial_\beta\mathbf{r}_0,\ \partial_t\mathbf{r}_0\|\mathbf{r}_0$ form an orthogonal basis in space and define tree unit vectors $\mathbf{e}_\alpha(\alpha,\beta)$, $\mathbf{e}_\beta(\alpha,\beta)$, $\mathbf{e}_t(\alpha,\beta)$ such that

$$\partial_\alpha\mathbf{r}_0=h_\alpha\mathbf{e}_\alpha,\quad \partial_\beta\mathbf{r}_0=h_\beta\mathbf{e}_\beta,\ \ \partial_t\mathbf{r}_0=h_t\mathbf{e}_t,\quad \mathbf{e}_\alpha\times\mathbf{e}_\beta=\mathbf{e}_t, \tag{50}$$

where $\mathbf{e}_\alpha,\ \mathbf{e}_\beta,\ \mathbf{e}_t$ are independent of t, and

$$h_t=\dot r_0,\ \ h_\alpha=r_0(1-\alpha^2)^{-1/2},\ \ h_\beta=r_0(1-\alpha^2)^{1/2} \tag{51}$$

are the metric elements defined by

$$dx^2+dy^2+dz^2=h_t^2dt^2+h_\alpha^2d\alpha^2+h_\beta^2d\beta^2. \tag{52}$$

Inserting Eqs. (47) and (48) into Eqs. (7) and (8) we obtain

$$\partial_t m_0(t)=h_t h_\alpha h_\beta=\dot r_0 r_0^2, \tag{53}$$

which gives for the shell mass for unit solid angle

$$m_0(t)=r_0^3(t)/3 \tag{54}$$

and

$$\partial_t[m_0(t)\partial_t r_0(t)]=(h_\alpha h_\beta)/r_0^4(t), \tag{55}$$

i.e.,

$$\partial_t[r_0^3(t)\partial_t r_0(t)]=3/r_0^2(t), \tag{56}$$

which gives for large t

$$r_0(t)\approx 3^{1/2}t^{1/3}. \tag{57}$$

Note that the kinetic energy decreases as $t^{-1/3}$. Equations (54) and (57) can be obtained directly from Eqs. (3) and (4) by simply assuming spherical symmetry as done in Eq. (2). However, the above derivation illustrates the procedure that will be used for the stability analysis of the spherical shell expansion.

B. Spherical shell expansion stability

We linearize Eqs. (7) and (8) around the spherical solution $r_0(t)$ given by Eq. (57) and write

$$\begin{aligned} \mathbf{r}(\alpha,\beta,t)&=\mathbf{r_0}(t)+\delta\mathbf{r}(\alpha,\beta,t)\\ &=\mathbf{r_0}(t)+\delta r_t(\alpha,\beta,t)\mathbf{e}_t+\delta r_\alpha(\alpha,\beta,t)\mathbf{e}_\alpha\\ &\quad+\delta r_\beta(\alpha,\beta,t)\mathbf{e}_\beta. \end{aligned} \tag{58}$$

Inserting Eq. (58) into the mass equation (7), we obtain

$$\begin{aligned} \partial_t\delta m=&(\partial_t\delta\mathbf{r}\cdot[\partial_\alpha\mathbf{r}\times\partial_\beta\mathbf{r}])+(\partial_\alpha\delta\mathbf{r}\cdot[\partial_\beta\mathbf{r}\times\partial_t\mathbf{r}])\\ &+(\partial_\beta\delta\mathbf{r}\cdot[\partial_t\mathbf{r}\times\partial_\alpha\mathbf{r}]), \end{aligned} \tag{59}$$

which, using Eq. (50), can be rewritten as

$$\partial_t\delta m=h_t h_\alpha h_\beta[h_t^{-1}\mathbf{e}_t\cdot\partial_t\delta\mathbf{r}+h_\alpha^{-1}\mathbf{e}_\alpha\cdot\partial_\alpha\delta\mathbf{r}+h_\beta^{-1}\mathbf{e}_\beta\cdot\partial_\beta\delta\mathbf{r}], \tag{60}$$

where the equilibrium mass variation $h_t h_\alpha h_\beta$ has been factorized out. As shown in Appendix B, Eq. (60) involves the divergence of the displacement vector $\delta\mathbf{r}$ and can be rewritten in the form

$$\partial_t \delta m = \partial_t(\delta r_t h_\alpha h_\beta) + \partial_\alpha(\delta r_\alpha h_\beta h_t) + \partial_\beta(\delta r_\beta h_t h_\alpha). \tag{61}$$

From $|\mathbf{r}(\alpha,\beta,t)|^2 \approx r_0(t)^2 + 2 r_0(t)\mathbf{e}_t \cdot \delta\mathbf{r}(\alpha,\beta,t) = r_0(t)^2 + 2 r_0(t)\delta r_t(\alpha,\beta,t)$ we obtain for the pressure term $1/r^4$

$$r^{-4}(\alpha,\beta,t) \approx r_0(t)^{-4}[1 - 4\delta r_t(\alpha,\beta,t)/r_0(t)]. \tag{62}$$

Then linearizing Eq. (8) and recalling that, since the equilibrium expansion is uniform, the unit vectors $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$, $\mathbf{e}_t$ are independent of t, we use Eqs. (B2) and (B3) of Appendix B to obtain

$$\begin{aligned}\partial_t(m_0\partial_t \delta r_t + \delta m\, \partial_t r_0) = r_0(t)^{-4}\{&[\partial_\alpha(h_\beta \delta r_\alpha) + \partial_\beta(h_\alpha \delta r_\beta)\\ &+ h_t^{-1}\delta r_t \partial_t(h_\alpha h_\beta)]\\ &- 4 h_\alpha h_\beta \delta r_t / r_0(t)\},\end{aligned} \tag{63}$$

where we recall that $h_\alpha h_\beta = r_0^2(t)$,

$$\partial_t(m_0\partial_t \delta r_\alpha) = r_0(t)^{-4}(-h_\beta \partial_\alpha \delta r_t + h_\beta \delta r_\alpha h_t^{-1}\partial_t h_\alpha) \tag{64}$$

and

$$\partial_t(m_0\partial_t \delta r_\beta) = r_0(t)^{-4}(-h_\alpha \partial_\beta \delta r_t + h_\alpha \delta r_\beta h_t^{-1}\partial_t h_\beta). \tag{65}$$

Equations (61), (63), (64) and (65) form our basic system of linearized equations. Since the zero order shell expansion is spherically symmetric we can write the perturbations in terms of a basis of eigenmodes with angular momentum l and azimuthal number s. This decomposition into spherical harmonics in the Lagrangian variables α and β is better performed by reverting to the variable θ, such that $(1-\alpha^2)^{1/2} = \sin\theta$ and $\partial_\alpha = (1/\sin\theta)\partial_\theta$, and by appropriately rewriting Eqs. (61), (63), (64), and (65) as detailed in Appendix B.

We expand $\delta\mathbf{r}(\alpha,\beta,t)$ in spherical harmonics according to

$$\delta r_t(\alpha,\beta,t) = \sum_{l,s} r_{\parallel l,s}(t) Y_{l,s}(\theta)\exp(is\beta), \tag{66}$$

$$\delta r_\alpha(\alpha,\beta,t) = \sum_{l,s} r_{\perp l,s}(t)\partial_0 Y_{l,s}(\theta)\exp(is\beta), \tag{67}$$

$$\delta r_\beta(\alpha,\beta,t) = -\sum_{l,s} r_{\perp l,s}(t)\frac{is Y_{l,s}(\theta)}{\sin\theta}\exp(is\beta), \tag{68}$$

where a standard notation for the spherical harmonic functions $Y_{l,s}$ has been adopted. Here $r_{\parallel l,s}(t)$ and $r_{\perp l,s}(t)$ are the amplitudes of the radial and of the tangential displacements, respectively. As shown in Appendix B they are related by the differential equation

$$\partial_t[m_0\partial_t r_{\perp l,s}(t)] = r_0(t)^{-3}[r_{\perp l,s}(t) + r_{\parallel l,s}(t)]. \tag{69}$$

Then, after expanding the perturbed mass term in spherical harmonics,

$$\delta m(\alpha,\beta,t) = \sum_{l,s} \mu_{l,s}(t) Y_{l,s}(\theta)\exp(is\beta), \tag{70}$$

we find that Eqs. (61) and (63) read, see Appendix B,

$$\partial_t \mu_{l,s} = \partial_t[r_0^2 r_{\parallel l,s}(t)] + [l(l+1)/2](\partial_t r_0^2) r_{\perp l,s}(t), \tag{71}$$

$$\begin{aligned}\partial_t[m_0\partial_t r_{\parallel l,s}(t) + \mu_{l,s}\partial_t r_0] = \frac{1}{r_0(t)^3}[&l(l+1) r_{\perp l,s}(t)\\ &- 2 r_{\parallel l,s}(t)].\end{aligned} \tag{72}$$

Note that Eqs. (69), (71), and (72) are independent of the azimuthal mode number s as a consequence of the spherical symmetry of the zero order expansion. Thus in this linear analysis we can set $s=0$ (and $\delta r_\beta = 0$) without loss of generality. Note in addition that for $l=0$ Eq. (69) decouples and $\delta r_\alpha \equiv 0$.

Similarly to the cylindrical case, the solutions of Eqs. (69), (71), and (72) are of the form

$$\mu_l(t) = \hat{\mu}_l t^{\gamma+2/3}, \quad r_{\parallel l}(t) = \hat{r}_{\parallel l} t^{\gamma}, \quad r_{\perp l}(t) = \hat{r}_{\perp l} t^{\gamma}, \tag{73}$$

where the exponent $\gamma = \gamma(l)$ is given by the solutions of the fourth order polynomial

$$(2 + 9\gamma + 9\gamma^2)^2 + 2l(l+1) = 0. \tag{74}$$

Stable solutions correspond to Re $\gamma \leqslant 1/3$. We find stability for $l \leqslant 8$ (actually, all roots have Re $\gamma < 0$ for $l \leqslant 2$). The fifth solution of Eqs. (69), (71), and (72) corresponds to $\gamma = 1/3$ and has $\delta r_t = 0$ for all l, which implies that the distance from the origin is not changed. As in the case of the planar and of the cylindrical configurations, this fifth solution corresponds to a relabeling of the Lagrangian variables. For large values of l Eq. (74) reduces to $\gamma^2 = \pm i(\sqrt{2}/9) l$ which is again a modification of Ott's result [22]. These results show that the effect of the mass accretion and of the decrease of the pressure stabilizes the long wavelength Rayleigh-Taylor modes of a spherical foil somewhat more efficiently than in the case of a cylindrical foil.

V. CONCLUSIONS

In the present paper we have studied the linear stability of an infinitely thin shell expanding in an ambient plasma and accreting all the mass it sweeps through (the snowplow approximation) under the push of the ponderomotive force of the electromagnetic fields trapped by the shell. We have developed a general formalism based on the use of Lagrangian variables that applies to the expansion of shells of arbitrary shape and have considered in particular the evolution of a planar, of a cylindrical, and of a spherical shell.

The accreting mass and the decreasing pressure lead to different expansion laws in these three different configurations and to different stability properties. In particular, in the planar case the development of the Rayleigh-Taylor instability is only slowed down by these effects but the system remains unstable both at large and at small wavelengths. On the contrary, in the case of cylindrical and spherical expansion long wavelength modes are stabilized while short wavelength modes, which are not physically described properly within the thin shell model approximation, remain unstable (with an oscillation frequency of the order of the growth rate).

The same formalism can be applied to the study of the shell expansion in a weakly inhomogeneous medium. The shell expands faster in the direction where the ambient density is lower. This results in the deformation of the bubble and of its acceleration against the density gradient.

APPENDIX A

We separate the perturbations in Eq. (21) into even and odd perturbations under the transformation given by Eq. (20). Thus, for even modes the coefficients w_k are real, $w_k = w_{k,e}$, and for odd modes they are imaginary, $w_k = i w_{k,o}$. The harmonics k and $-k$ are coupled and for a chosen $|k|$ $(=k)$, we obtain

$$x(\alpha,t) = x_0 + \delta x_{k,e}(\alpha,t) + \delta x_{k,o}(\alpha,t),$$
$$y(\alpha,t) = \alpha + \delta y_{k,e}(\alpha,t) + \delta y_{k,o}(\alpha,t), \tag{A1}$$

where the indices e,o denote even and odd modes, respectively,

$$\delta x_{k,e}(\alpha,t) = (w_{k,e} + w_{-k,e})\cos(k\alpha),$$
$$\delta y_{k,e}(\alpha,t) = (w_{k,e} - w_{-k,e})\sin(k\alpha) \tag{A2}$$

and

$$\delta x_{\alpha,o}(\alpha,t) = -(w_{k,o} - w_{-k,o})\sin(k\alpha),$$
$$\delta y_{k,o}(\alpha,t) = (w_{k,o} + w_{-k,o})\cos(k\alpha). \tag{A3}$$

Translations along α mix even and odd modes. Since the equilibrium and Eqs. (12) and (13) are translationally invariant, this implies that even and odd modes are degenerate.

APPENDIX B

1. Derivatives of the displacement vector

From Eqs. (47) and (48) and the following equations we obtain the identities

$$h_t^{-1}\mathbf{e}_t \cdot \partial_t \delta\mathbf{r} = h_t^{-1}\partial_t \delta r_t, \tag{B1}$$

$$h_\alpha^{-1}\mathbf{e}_\alpha \cdot \partial_\alpha \delta\mathbf{r} = h_\alpha^{-1}\partial_\alpha \delta r_\alpha + h_t^{-1}h_\alpha^{-1}\delta r_t \partial_t h_\alpha, \tag{B2}$$

$$h_\beta^{-1}\mathbf{e}_\beta \cdot \partial_\beta \delta\mathbf{r} = h_\beta^{-1}\partial_\beta \delta r_\beta + h_\alpha^{-1}h_\beta^{-1}\delta r_\alpha \partial_\alpha h_\beta + h_t^{-1}h_\beta^{-1}\delta r_t \partial_t h_\beta, \tag{B3}$$

where we have used $\partial_\alpha h_t = 0$ and the symmetry property $\partial_\beta(h_t, h_\alpha, h_\beta) = 0$. For reference the explicit calculation of Eq. (B1) is reported as

$$\begin{aligned}\mathbf{e}_t \cdot \partial_t \delta\mathbf{r} &= \mathbf{e}_t \cdot \partial_t(\delta r_t \mathbf{e}_t + \delta r_\alpha \mathbf{e}_\alpha + \delta r_\beta \mathbf{e}_\beta)\\ &= \partial_t \delta r_t + (\delta r_\alpha / h_\alpha)\mathbf{e}_t \cdot \partial_t \partial_\alpha \mathbf{r}_0\\ &\quad + (\delta r_\beta / h_\beta)\mathbf{e}_t \cdot \partial_t \partial_\beta \mathbf{r}_0\\ &= \partial_t \delta r_t + (\delta r_\alpha / h_\alpha)(\mathbf{e}_t \cdot \mathbf{e}_t)\partial_\alpha h_t\\ &\quad + (\delta r_\beta / h_\beta)(\mathbf{e}_t \cdot \mathbf{e}_t)\partial_\beta h_t.\end{aligned}$$

Summing Eqs. (B1)–(B3) we obtain

$$\begin{aligned}&h_t^{-1}\mathbf{e}_t \cdot \partial_t \delta\mathbf{r} + h_\alpha^{-1}\mathbf{e}_\alpha \cdot \partial_\alpha \delta\mathbf{r} + h_\beta^{-1}\mathbf{e}_\beta \cdot \partial_\beta \delta\mathbf{r}\\ &\quad = [h_t h_\alpha h_\beta]^{-1}[\partial_t(\delta r_t h_\alpha h_\beta)\\ &\qquad + \partial_\alpha(\delta r_\alpha h_\beta h_t) + \partial_\beta(\delta r_\beta h_t h_\alpha)]\\ &\quad \equiv \mathrm{div} \cdot \delta\mathbf{r},\end{aligned} \tag{B4}$$

where we used the definition of the divergence of a vector in curvilinear coordinates.

2. Expansion in spherical harmonics

If we reintroduce the angle θ, such that $(1-\alpha^2)^{1/2} = \sin\theta$ and $\partial_\alpha = (-1/\sin\theta)\partial_\theta$, and substitute the explicit values of the metric elements $h_\alpha = r_0(t)/\sin\theta$, $h_\beta = r_0(t)\sin\theta$, $h_t = \dot r_0(t)$, Eq. (61) reads

$$\partial_t \delta m = \partial_t(r_0^2 \delta r_t) + \left(\frac{\partial_t r_0^2}{2}\right)\left[\frac{-\partial_\theta(\sin\theta\,\delta r_\alpha)}{\sin\theta} + \frac{\partial_\beta \delta r_\beta}{\sin\theta}\right], \tag{B5}$$

and Eqs. (63)–(65) read

$$\partial_t(m_0 \partial_t \delta r_t + \delta m\, \partial_t r_0) = \frac{1}{r_0(t)^3}\left[\frac{-\partial_\theta(\sin\theta\,\delta r_\alpha)}{\sin\theta} + \frac{\partial_\beta \delta r_\beta}{\sin\theta} - 2\,\delta r_t\right], \tag{B6}$$

$$\partial_t(m_0 \partial_t \delta r_\alpha) = \frac{1}{r_0(t)^3}(\partial_\theta \delta r_t + \delta r_\alpha), \tag{B7}$$

$$\partial_t(m_0 \partial_t \delta r_\beta) = \frac{1}{r_0(t)^3}\left(-\frac{\partial_\beta \delta r_t}{\sin\theta} + \delta r_\beta\right). \tag{B8}$$

If we expand the radial displacement δr_t in spherical harmonics according to Eq. (66) and use Eqs. (64) and (65) as ‘‘inhomogeneous’’ equations for δr_α and δr_β to define the expansion δr_α and δr_β in spherical harmonics, we obtain Eqs. (67) and (68) together with the relationship between the amplitude of the tangential and of the radial displacement given by Eq. (69). Inserting Eqs. (67) and (68) into Eqs. (B6) and (B7) we find that the differential operator inside the square brackets reduces to the angular part of the Laplacian operator, i.e., for fixed l and s

$$-\frac{\partial_\theta(\sin\theta\,\delta r_\alpha)}{\sin\theta}+\frac{\partial_\beta \delta r_\beta}{\sin\theta}$$
$$=-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial_\theta}\sin\theta\frac{\partial}{\partial_\theta}-\frac{s^2}{\sin^2\theta}\right]Y_{l,s}(\theta)\exp(is\beta)r_{\perp l,s}(t)$$
$$=l(l+1)Y_{l,s}(\theta)\exp(is\beta)r_{\perp l,s}(t), \tag{B9}$$

independently of s.

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